

# Accelerated Charge Kerr-Schild Metrics in $D$ - Dimensions

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## Abstract

We consider the  $D$  dimensional Einstein Maxwell theory with a null fluid in the Kerr-Schild Geometry. We obtain a complete set of differential conditions that are necessary for finding solutions. We examine the case of vanishing pressure and cosmological constant in detail. For this specific case, we give the metric, the electromagnetic vector potential and the fluid energy density. This is, in fact, the generalization of the well known Bonnor-Vaidya solution to arbitrary  $D$  dimensions. We show that due to the acceleration of charged sources, there is an energy flux in  $D \geq 4$  dimensions and we give the explicit form of this energy flux formula.

# 1 Introduction

Radiation and hence energy loss due to the acceleration of an electron is a well known phenomenon in classical electromagnetism. An exact solution describing this phenomenon in General Relativity is due to Bonnor and Vaidya [1], where the metric is in Kerr-Schild form [2], [3], [4]. An acceleration parameter has also been considered in Robinson-Trautman metrics [5]-[7]. The energy loss formula turns out to be exactly the same as the one obtained from classical electromagnetism. When the acceleration vanishes, Bonnor-Vaidya metrics reduce to the Reissner-Nordström (RN) metric.

In this work our main motivation is to generalize Bonnor-Vaidya and Photon Rocket solutions of  $D = 4$  General Relativity. For this purpose we consider the  $D$ -dimensional Kerr-Schild metric with an appropriate vector potential and a fluid velocity vector, and we derive a complete set of conditions for the Einstein Maxwell theory with a null perfect fluid. We find the expressions for the pressure and the mass density of the fluid. We classify our solutions under some certain assumptions. The field equations are highly complex and with some nontrivial equation of state it is quite difficult to solve the corresponding equations. To this end we assume vanishing pressure and cosmological constant. We give the complete solution for arbitrary dimension  $D$ . This generalizes the Bonnor-Vaidya solution. We obtain the energy flux formula depending on the dimension  $D$ .

Our conventions are similar to the conventions of Hawking-Ellis [8]. This means that the Riemann tensor  $R^\alpha{}_{\beta\mu\nu}$ , Ricci tensor  $R_{\alpha\beta}$ , Ricci scalar  $R_s$  and the Einstein tensor  $G_{\mu\nu}$  are defined by

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\gamma{}_{\beta\nu} - \Gamma^\alpha{}_{\nu\gamma}\Gamma^\gamma{}_{\beta\mu}, \quad (1)$$

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}, \quad R_s = R^\alpha{}_\alpha, \quad G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R_s \quad (2)$$

Here  $g_{\mu\nu}$  is the  $D$  dimensional metric tensor with signature  $(-, +, +, \dots, +)$ . The source of the Einstein equations is composed of the electromagnetic field

with the vector potential field  $A_\mu$  and a perfect fluid with velocity vector field  $u^\mu$ , energy density  $\rho$  and pressure  $p$ . Their energy momentum tensors are respectively given by

$$T_{\mu\nu}^e = F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}F^2 g_{\mu\nu}, \quad F^2 \equiv F^{\alpha\beta} F_{\alpha\beta}, \quad (3)$$

$$T_{\mu\nu}^f = (p + \rho) u_\mu u_\nu + p g_{\mu\nu} \quad (4)$$

Then the Einstein equations are given by

$$G_{\mu\nu} = \kappa T_{\mu\nu} = \kappa (T_{\mu\nu}^e + T_{\mu\nu}^f) + \Lambda g_{\mu\nu}, \quad (5)$$

$$(p + \rho) u^\nu u^\mu{}_{;\nu} = -u^\nu (\rho u^\mu)_{;\nu} + p_{;\nu} (g^{\mu\nu} + u^\mu u^\nu) + F^\mu{}_\nu J^\nu, \quad (6)$$

$$F^{\mu\nu}{}_{;\nu} = J^\mu \quad (7)$$

In the next section, we develop the kinematics of a curve  $C$  in the  $D$ - dimensional Minkowski manifold  $\mathbf{M}_D$ . We construct solutions of the electromagnetic vector field due to the acceleration of charged particles in four and six dimensions. We then find the energy flux due to acceleration. In Section 3, we give a detailed study of Kerr-Schild Geometry under certain assumptions. In Section 4, we find the solution of the  $D$ - dimensional Einstein Maxwell field equations with a null fluid. We also obtain the generalization of Bonnor-Vaidya metrics to  $D$ - dimensions. We derive the energy flux formula depending on the dimension  $D$  and discuss its finiteness in Section 5 and state our conclusion in Section 6. Finally, for convenience to the reader, in an Appendix we give some well-known formulae that are needed in the text.

## 2 Radiation due to Acceleration: Maxwell Theory

Let  $z^\mu(\tau)$  describe a smooth curve  $C$  defined by  $z : I \subset \mathbf{R} \rightarrow \mathbf{M}_D$ . Here  $\tau \in I$ ,  $I$  is an interval on the real line and  $\mathbf{M}_D$  is the  $D$ - dimensional Minkowski manifold. From an arbitrary point  $x^\mu$  outside the curve, there are two null lines intersecting the curve  $C$ . These points are called the retarded and advanced times. Let  $\Omega$  be the distance between the points  $x^\mu$  and  $z^\mu(\tau)$ , so by definition it is given by

$$\Omega = \eta_{\mu\nu} (x^\mu - z^\mu(\tau)) (x^\nu - z^\nu(\tau)), \quad (8)$$

Hence  $\Omega = 0$  for two values of  $\tau$  for a non-space-like curve. Let us call these as,  $\tau_0$  (retarded) and  $\tau_1$  (advanced). We shall focus ourselves to the retarded case only. The main reason for this is that the Green's function for the vector potential chooses this point on the curve  $C$  ([9], [10]).

If we differentiate  $\Omega$  with respect to  $x^\mu$  and let  $\tau = \tau_0$ , then we get

$$\lambda_\mu \equiv \tau_{,\mu} = \frac{x_\mu - z_\mu(\tau_0)}{R}, \quad (9)$$

where  $\lambda_\mu$  is a null vector and  $R$  is the retarded distance defined by

$$R \equiv \dot{z}^\mu(\tau_0) (x_\mu - z_\mu(\tau_0)). \quad (10)$$

Here a dot over a letter denotes differentiation with respect to  $\tau_0$ . We now list some properties of  $R$  and  $\lambda_\mu$ .

$$\lambda_{\mu,\nu} = \frac{1}{R} [\eta_{\mu\nu} - \dot{z}_\mu \lambda_\nu - \dot{z}_\nu \lambda_\mu - (A - \epsilon) \lambda_\mu \lambda_\nu], \quad (11)$$

$$R_{,\mu} = (A - \epsilon) \lambda_\mu + \dot{z}_\mu \quad (12)$$

where

$$A = \ddot{z}^\mu (x_\mu - z_\mu), \quad \dot{z}^\mu \dot{z}_\mu = \epsilon = 0, \pm 1 \quad (13)$$

Here  $\epsilon = -1$  for time-like curves and vanishes for null curves. Furthermore we have

$$\lambda_\mu \dot{z}^\mu = 1, \quad \lambda^\mu R_{,\mu} = 1 \quad (14)$$

Let  $a = \frac{A}{R}$ . Then it is easy to prove that

$$a_{,\mu} \lambda^\mu = 0 \quad (15)$$

Similar to  $a$ , we have other scalars satisfying the same property (15) obeyed by  $a$ .

**Lemma 1.** *Let*

$$a_k = \lambda_\mu \frac{d^k \ddot{z}^\mu}{d\tau_0^k}, \quad k = 1, 2, \dots, n \quad (16)$$

*then*

$$a_{k,\alpha} \lambda^\alpha = 0, \quad (17)$$

*for all  $k$ . Furthermore if  $A_k = R a_k$  is constant for a fixed  $k$  at all points then  $A_i = 0$  for all  $i \geq k$  and  $d^m \ddot{z}^\mu / d\tau_0^m = 0$  for all  $m \geq k$ . Here  $n$  is an arbitrary positive integer.*

**Proof.** The proof of this Lemma depends on the following formula for the derivative of  $A_k$

$$A_{k,\alpha} = \frac{d^k \ddot{z}_\alpha}{d\tau_0^k} + [A_{k+1} - (d^k \ddot{z}^\beta / d\tau_0^k) \dot{z}_\beta] \lambda_\alpha. \quad (18)$$

In the sequel, for the sake of simplicity, we shall use  $\tau$  instead of  $\tau_0$ . First part of the Lemma can be proved by contracting the above formula (18) by  $\lambda^\alpha$ . One obtains

$$\lambda^\alpha A_{k,\alpha} = a_k. \quad (19)$$

This implies that  $\lambda^\alpha a_{k,\alpha} = 0$ . For the second part of the Lemma contracting the above formula (18) by  $\dot{z}^\alpha$  one obtains

$$A_{k,\alpha} \dot{z}^\alpha = A_{k+1}, \quad (20)$$

which implies that  $A_m = 0$  for  $m > k$ . From (19) we have also  $A_k = 0$ . With these results (18) reduces to

$$\frac{d^k \ddot{z}_\alpha}{d\tau^k} = \mu \lambda_\alpha, \quad \mu = (d^k \ddot{z}^\beta / d\tau^k) \dot{z}_\beta. \quad (21)$$

Differentiating this equation with respect to  $x^\beta$  and contracting the resulting equation by  $\lambda^\beta$  we get  $\mu = 0$ . This completes the proof of the Lemma.

In (16)  $n$  depends on the dimension  $D$  of the manifold  $\mathbf{M}_D$ . The scalars  $(a, a_k)$ , are related to the curvature scalars of the curve  $C$  in  $\mathbf{M}_D$ . The number of such scalars is  $D - 1$  [11], [12]. Hence we let  $n = D - 1$ .

Before moving on to the main subject of this work, i.e. examining the Einstein-Maxwell theory within the Kerr-Schild geometry, we briefly study the Maxwell theory in flat (even)  $D$ -dimensional Minkowski space. This should at least serve the reader as a reminder of the well-known basics regarding the usual  $D = 4$  Maxwell theory.

By using the above curve  $C$  and its kinematics, we can construct divergence free (Lorentz gauge) vector fields  $A_\alpha$  satisfying the wave equation  $\eta^{\mu\nu} \partial_\mu \partial_\nu A_\alpha = 0$  outside the curve  $C$  in any even dimension  $D$ . For instance in the cases  $D = 4$  and  $D = 6$  we have respectively [13], [14]

$$A_\mu = \begin{cases} \frac{e}{4\pi} \frac{\dot{z}_\mu}{R}, & (D = 4) \\ \frac{e}{4\pi^2} \left[ \frac{\ddot{z}_\mu - a \dot{z}_\mu}{R^2} + \epsilon \frac{\dot{z}_\mu}{R^3} \right] & (D = 6) \end{cases} \quad (22)$$

These vector fields represent respectively the electromagnetic vector potentials of an accelerated charge in four and six dimensional Maxwell theory of

electromagnetism. The flux of electromagnetic energy is then given by [9]

$$dE = - \int_S \dot{z}_\mu T^{\mu\nu} dS_\nu \quad (23)$$

where  $T_{\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} F^2 \eta_{\mu\nu}$  is the Maxwell energy momentum tensor,  $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$  is the electromagnetic field tensor and  $F^2 = F^{\alpha\beta} F_{\alpha\beta}$ . The surface element  $dS_\mu$  on  $S$  is given by

$$dS_\mu = n_\mu R d\tau d\Omega \quad (24)$$

where  $n_\nu$  is orthogonal to the velocity vector field  $\dot{z}_\mu$  which is defined through

$$\lambda_\mu = \epsilon \dot{z}_\mu + \epsilon_1 \frac{n_\mu}{R}, \quad n^\mu n_\mu = -\epsilon R^2 \quad (25)$$

Here  $\epsilon_1 = \pm 1$ . For the remaining part of this section we shall assume  $\epsilon = -1$  ( $C$  is a time-like curve). One can consider  $S$  in the rest frame as a sphere of radius  $R$ . Here  $d\Omega$  is the solid angle. Letting  $dE/d\tau = N_e$  [1], then we have

$$N_e = - \int_S \dot{z}_\mu T^{\mu\nu} n_\nu R d\Omega \quad (26)$$

At very large values of  $R$  we get for  $D = 4$

$$N_e = (e/4\pi)^2 \epsilon_1 \int (-\ddot{z}_\mu \ddot{z}^\mu + a^2) d\Omega \quad (27)$$

$$= -\epsilon_1 (e/4\pi)^2 (\ddot{z}^\mu \ddot{z}_\mu) \int (1 - \cos^2 \theta) \sin \theta d\theta d\phi, \quad (28)$$

$$= -\epsilon_1 (e^2/4\pi) \frac{2}{3} (\ddot{z}^\mu \ddot{z}_\mu) \quad (29)$$

and for  $D = 6$  at very large values of  $R$  we have

$$N_e = - \int_S \dot{z}_\mu T^{\mu\nu} n_\nu R^3 d\Omega, \quad (30)$$

$$= -\left(\frac{e}{4\pi^2}\right)^2 \epsilon_1 \int \xi_\mu \xi^\mu d\Omega \quad (31)$$

where

$$\xi^\mu = \frac{d^3 z^\mu}{d\tau^3} - 3a \frac{d^2 z^\mu}{d\tau^2} + (-a_1 + 3a^2) \frac{dz^\mu}{d\tau} \quad (32)$$

so that  $\lambda_\mu \xi^\mu = 0$ . For a charge  $e$  with acceleration  $|\ddot{z}_\mu| = \kappa_1$  we have (for  $D = 6$  case)

$$N_e = -\left(\frac{e}{4\pi^2}\right)^2 \frac{32\pi^2 \epsilon_1}{15} (\dot{\kappa}_1^2 + \kappa_1^2 \kappa_2^2 + \frac{9}{7} \kappa_1^4) \quad (33)$$

**Remark 1.** To be compatible with the classical results [9], [10] we take  $\epsilon_1 = -1$ . We also conjecture that the sign of the energy flux will be the same for all even dimensions.

### 3 Accelerated Kerr-Schild Metrics in $D$ - Dimensions

We now consider the Einstein-Maxwell field equations with a perfect fluid distribution in  $D$ -dimensions. Here we make some assumptions. First of all, we assume that the metric of the  $D$ -dimensional space-time is the Kerr-Schild metric. Furthermore, we take the null vector  $\lambda_\mu$  in the metric as the same null vector defined in (9). With these assumptions the Ricci tensor takes a special form.

**Proposition 2.** *Let  $g_{\mu\nu} = \eta_{\mu\nu} - 2V\lambda_\mu\lambda_\nu$  and  $\lambda_\mu$  be the null vector defined in (9) and let  $V$  be a differentiable function, then the Ricci tensor and the Ricci scalar are given by*

$$R^\alpha{}_\beta = \zeta_\beta \lambda^\alpha + \zeta^\alpha \lambda_\beta + r\delta^\alpha{}_\beta + q\lambda_\beta \lambda^\alpha, \quad (34)$$

$$R_s = -2\lambda^\alpha K_{,\alpha} - 4\theta K - \frac{2V}{R^2}(D-2)(D-3), \quad (35)$$



where

$$r = \frac{2(-D+3)V}{R^2} - \frac{2K}{R}, \quad (36)$$

$$q = \eta^{\alpha\beta} V_{,\alpha\beta} + \epsilon r + \frac{2A}{R}(K + \theta V) - \frac{4}{R}(\dot{z}^\mu V_{,\mu} + AK - \epsilon K), \quad (37)$$

$$\zeta_\alpha = -K_{,\alpha} - \frac{D-4}{R}V_{,\alpha} + \frac{2V}{R^2}(D-3)\dot{z}_\alpha, \quad (38)$$

where  $K \equiv \lambda^\alpha V_{,\alpha}$  and  $\theta \equiv \lambda^\alpha_{,\alpha} = \frac{D-2}{R}$ .

Let us further assume that the electromagnetic vector potential  $A_\mu$  is given by  $A_\mu = H \lambda_\mu$  where  $H$  is a differentiable function. Let  $p$  and  $\rho$  be the pressure and the energy density of a perfect fluid distribution with the velocity vector field  $\lambda_\mu$ . Then the difference tensor  $\mathcal{T}^\mu{}_\nu = G^\mu{}_\nu - \kappa T^\mu{}_\nu$  is given by the following proposition:

**Proposition 3.** *Let  $g_{\mu\nu} = \eta_{\mu\nu} - 2V\lambda_\mu\lambda_\nu$ ,  $A_\mu = H\lambda_\mu$ , where  $\lambda_\mu$  is given in (9),  $V$  and  $H$  be differentiable functions. Let  $p$  and  $\rho$  be the pressure and energy density of a perfect fluid with velocity vector field  $\lambda_\mu$ . Then the difference tensor becomes*

$$\mathcal{T}^\alpha{}_\beta = \lambda^\alpha W_\beta + \lambda_\beta W^\alpha + \mathcal{P} \delta^\alpha{}_\beta + \mathcal{Q} \lambda^\alpha \lambda_\beta \quad (39)$$

where

$$\mathcal{P} = r - \frac{1}{2}R_s - \frac{1}{2}\kappa(\lambda^\mu H_{,\mu})^2 - (\kappa p + \Lambda), \quad (40)$$

$$\mathcal{Q} = q - \kappa(p + \rho) - \kappa(\eta^{\alpha\beta} H_{,\alpha} H_{,\beta}), \quad (41)$$

$$W_\alpha = \zeta_\alpha + \kappa(\lambda^\mu H_{,\mu})H_{,\alpha}, \quad (42)$$

where  $\zeta_\alpha$ ,  $r$ ,  $R_s$  and  $q$  are given in (38), (36), (35) and (37) respectively.

The vanishing of the difference tensor  $\mathcal{T}^\alpha{}_\beta$  in Proposition 3 implies that  $\mathcal{P} = 0$  and  $W_\alpha = -\frac{\mathcal{Q}}{2}\lambda_\alpha$ . Hence the following Corollary gives a set of

equations that are equivalent to the Einstein equations under the assumptions of Proposition 3.

**Corollary 4.** *With the assumptions of Proposition 3, the Einstein equations (5) imply that*

$$\kappa p + \Lambda = \frac{1}{2}\kappa(\lambda^\alpha H_{,\alpha})^2 + \frac{D-2}{R}K + \frac{(D-2)(D-3)}{R^2}V, \quad (43)$$

$$\kappa(p + \rho) = q - \kappa\eta^{\alpha\beta}H_{,\alpha}H_{,\beta} + 2w \quad (44)$$

$$W_\alpha = w\lambda_\alpha \quad (45)$$

where  $w = W^\alpha \dot{z}_\alpha$  and  $\Lambda$  is the cosmological constant. Here  $q$  and  $W_\alpha$  are respectively given in (37) and (42).

We shall now assume that the functions  $V$  and  $H$  depend on  $R$  and on some  $R$ -independent functions  $c_i$ , ( $i = 1, 2, \dots$ ) such that

$$c_{i,\alpha}\lambda^\alpha = 0, \quad (46)$$

for all  $i$ . It is clear that due to the property (15) and (17) of  $a_k$ 's, all of these functions ( $c_i$ ) are functions of the scalars  $a$  and  $a_k$ , ( $k = 1, 2, \dots, D-1$ ).

Notice that there are in fact  $D+2$  equations contained in (43)-(45). In particular using the vector equation (45) we can produce other scalar equations by contracting it with the vectors  $\lambda^\alpha$  and  $\dot{z}^\alpha$ . We summarize the results in the following Proposition.

**Proposition 5.** *Let  $V$  and  $H$  depend on  $R$  and functions  $c_i$ , ( $i = 1, 2, \dots$ ) that satisfy (46), then the Einstein equations given in Proposition 4 reduce to the following set of equations*

$$\kappa p + \Lambda = \frac{1}{2}V'' + \frac{3D-8}{2R}V' + \frac{(D-3)^2}{R^2}V, \quad (47)$$

$$\kappa(H')^2 = V'' + \frac{D-4}{R}V' - \frac{2V}{R^2}(D-3), \quad (48)$$

$$\kappa(p + \rho) = q - \kappa \eta^{\alpha\beta} H_{,\alpha} H_{,\beta} + 2 \left[ \frac{2(A - \epsilon)(D - 3)V}{R^2} - \sum_{i=1} (w_i c_{i,\alpha} \dot{z}^\alpha) \right], \quad (49)$$

$$\sum_{i=1} w_i c_{i,\alpha} = \left[ \sum_{i=1} (w_i c_{i,\beta} \dot{z}^\beta) \right] \lambda_\alpha, \quad (50)$$

where

$$w_i = V'_{,c_i} + \frac{D-4}{R} V_{,c_i} - \kappa H' H_{,c_i}, \quad (51)$$

and prime over a letter denotes partial differentiation with respect to  $R$ . Eq. (6) is satisfied identically with the electromagnetic current vector

$$J_\mu = \sum_{i=1} \left[ \frac{1}{R} (4 - D) H_{,c_i} - (H')_{,c_i} \right] c_{i,\mu} + \left[ -H'' + \frac{1}{R} (2 - D) H' \right] \dot{z}_\mu + \left[ \sum_{i=1} \left( 2(H')_{,c_i} (c_{i,\alpha} \dot{z}^\alpha) + H_{,c_i} (c_{i,\alpha})^{,\alpha} - \frac{2}{R} H_{,c_i} (c_{i,\alpha} \dot{z}^\alpha) \right) + A H'' \right] \lambda_\mu \quad (52)$$

Notice that (48) is obtained by contracting (45) with the vector  $\lambda^\alpha$ . The above equations can be described as follows. The equations (47) and (49) define the pressure and mass density of the perfect fluid distribution with null velocity  $\lambda_\mu$ . Eq.(48) gives a relation between the electromagnetic and gravitational potentials  $H$  and  $V$ . Since this relation is quite simple, given one of them one can easily solve the other one. Equation (50) implies that there are some functions  $c_i$  ( $i = 1, 2, \dots$ ) where this equation is satisfied. The functions  $c_i$  ( $i = 1, 2, \dots$ ) arise as integration constants (with respect to the variable  $R$ ) while determining the  $R$  dependence of the functions  $V$  and  $H$ . Assuming the existence of such  $c_i$ 's the above equations give the most general solution of the Einstein-Maxwell field equations with a null perfect fluid distribution under the assumptions of Proposition 3.

## 4 Null-Dust Solutions in $D$ -Dimensions

In this Section we give a class of new exact solutions in the Kerr-Schild geometry. Assuming that the null fluid has no pressure, the cosmological constant vanishes and using Proposition 5, we have the following result:

**Theorem 6.** *Let  $p = \Lambda = 0$ . Then*

$$V = \begin{cases} \frac{\kappa e^2 (D-3)}{2(D-2)} R^{-2D+6} + m R^{-D+3} & (D \geq 4) \\ -\frac{\kappa}{2} e^2 \ln R + m & (D = 3) \end{cases} \quad (53)$$

$$H = \begin{cases} c + \epsilon e R^{-D+3} & (D \geq 4) \\ c + \epsilon e \ln R & (D = 3) \end{cases} \quad (54)$$

where for  $D > 3$

$$\begin{aligned} \rho = & -(c_{,\alpha} c^\alpha) - \epsilon (3-D) e R^{3-D} (c^{\alpha}{}_{,\alpha}) - \epsilon 2(3-D)(2-D) e (c_{,\alpha} \dot{z}^\alpha) R^{2-D} \\ & + \frac{1}{\kappa} a(2-D)(1-D) M R^{2-D} + (2-D)(3-D) a e^2 R^{5-2D} \\ & - \epsilon (2-D)(1-D)(3-D) a c e R^{2-D} + \frac{1}{\kappa} \dot{M} (2-D) R^{2-D} \\ & - \epsilon (3-D)(2-D) c \dot{e} R^{2-D} + (3-D) e \dot{e} R^{5-2D}, \end{aligned} \quad (55)$$

$$\begin{aligned} J_\mu = & \frac{1}{R} (4-D) c_{,\mu} + [c^{\alpha}{}_{,\alpha} - \frac{2}{R} (c_{,\alpha} \dot{z}^\alpha) \\ & + \epsilon (3-D) \dot{e} R^{2-D} + \epsilon (3-D)(2-D) e a R^{2-D}] \lambda_\mu \end{aligned} \quad (56)$$

and for  $D = 3$

$$\begin{aligned} \rho = & -(c_{,\alpha} c^\alpha) - \epsilon e (c_{,\alpha}{}^{,\alpha}) + \epsilon \frac{2e}{R} (c_{,\alpha} \dot{z}^\alpha) - \frac{a e^2}{2R} \\ & + \frac{2Ma}{\kappa R} - \frac{a e^2}{R} \ln R - \frac{\dot{M}}{\kappa R} + \frac{e \dot{e}}{R} + \epsilon \frac{c \dot{e}}{R} \\ & + \frac{e \dot{e}}{R} \ln R - 2\epsilon e c \frac{a}{R}, \end{aligned} \quad (57)$$

$$J_\mu = \frac{1}{R} c_{,\mu} + [c^{\alpha}{}_{,\alpha} - \frac{2}{R} (c_{,\alpha} \dot{z}^\alpha) + \frac{\epsilon}{R} \dot{e} - \epsilon e \frac{a}{R}] \lambda_\mu \quad (58)$$

Here  $M = m + \epsilon \kappa(3 - D)ec$  for  $D \geq 4$  and  $M = m + \frac{\kappa}{2}e^2 + \epsilon \kappa ec$  for  $D = 3$ . In all cases  $e$  is assumed to be a function of  $\tau$  only but the functions  $m$  and  $c$  which are related through the arbitrary function  $M(\tau)$  (depends on  $\tau$  only) do depend on the scalars  $a$  and  $a_k$ , ( $k \geq 1$ ).

Equation (47) with zero pressure and (48) determine the  $R$  dependence of the potentials  $V$  and  $H$  completely. Using Proposition 5 we have chosen the integration constants ( $R$  independent functions) as the functions  $c_i$  ( $i = 1, 2, 3$ ) so that  $c_1 = m$ ,  $c_2 = e$  and  $c_3 = c$  and

$$c = c(\tau, a, a_k), \quad e = e(\tau),$$

$$m = \begin{cases} M(\tau) + \epsilon \kappa(D - 3)ec & (D \geq 4) \\ M(\tau) - \frac{\kappa}{2}e^2 - \epsilon \kappa ec & (D = 3) \end{cases}$$

where  $a_k$ 's are defined in Lemma 1.

**Remark 2.** The curve  $C$  is a geodesic of the curved geometry with the metric  $g_{\mu\nu} = \eta_{\mu\nu} - 2V\lambda_\mu\lambda_\nu$  if and only if it is a straight line in  $\mathbf{M}_D$ .

**Remark 3.** When  $D = 4$ , we obtain the Bonnor-Vaidya solutions with one essential difference. In the Bonnor-Vaidya solutions the parameters  $m$  and  $c$  (which are related through (4)) depend upon  $\tau$  and  $a$  only. In our solution, these parameters depend not only on  $\tau$  and  $a$  but also on all other scalars  $a_k$ , ( $k \geq 1$ ).

**Remark 4.** The electromagnetic vector potentials in the flat space (see Section 2) and in the curved space (see Section 3) are different. It is known that in four dimensions they are gauge equivalent [1]. It is this equivalence that led Bonnor and Vaidya to choose the function  $c(\tau, a, a_k) = -ea$ . All other choices of  $c$  do not have flat space limits. In higher dimensions ( $D > 4$ ) such an equivalence can not be established. As an example, let

$$A_f^\mu = \frac{\ddot{z}^\mu - a\dot{z}^\mu}{R^2} + \epsilon \frac{\dot{z}^\mu}{R^3}$$

be the vector potential (22) in 6-dimensional flat space. Let  $A_\mu = H\lambda_\mu = \Phi_{,\mu} + A_{f\mu}$ , where  $\Phi$  is the gauge potential to be determined. Here  $A_\mu$  is the vector potential (54) in 6-dimensional curved spacetime. Except for the static case, it is a simple calculation to show that there doesn't exist any  $\Phi$  to establish such a gauge equivalence. Hence our higher dimensional solutions have no flat space counterparts. In the static case (the curve  $C$  is a straight line) our solutions are, for all  $D$ , gauge equivalent to the Tangherlini [15] solutions (see also [16]). Because of this reason the energy flux expressions in 6-dimensions (see the next section) in flat and curved spacetimes are different. To obtain gauge equivalent solutions we conjecture that the Kerr-Schild ansatz has to be abandoned.

**Remark 5.** (a) It is easy to prove that  $\rho = 0$  only when the curve  $C$  is a straight line in  $\mathbf{M}_D$  (static case). This means that there are no accelerated vacuum and electro-vacuum solutions.

(b) We can have pure fluid solutions when  $e = c = 0$ . In this case we have

$$\begin{aligned} V &= mR^{3-D}, \\ \rho &= \frac{2-D}{\kappa} [a(1-D)M + \dot{M}] R^{2-D} \end{aligned} \quad (59)$$

for  $D \geq 4$  and

$$\begin{aligned} V &= m, \\ \rho &= \frac{2Ma - \dot{M}}{\kappa R} \end{aligned} \quad (60)$$

for  $D = 3$ . Such solutions are usually called as the *Photon Rocket* solutions [17]-[20]. We give here the  $D$  dimensional generalizations of this type of metrics as well.

## 5 Radiation due to Acceleration

In this Section we give a detailed analysis of the energy flux due to the acceleration of charged sources in the case of the solution given in Theorem 6. For the  $D > 3$  case, the solution described by the functions  $c$ ,  $e$ , and  $M$  give the energy density given in (55). Remember that at this point  $c = c(\tau, a, a_k)$  and arbitrary. Choosing  $e = \text{constant}$ ,  $c = -ea$  as was done by Bonnor and Vaidya [1], the expression for  $\rho$  simplifies and one gets

$$\begin{aligned} \rho = & \frac{1}{\kappa}(2-D)\dot{M}R^{2-D} + \frac{a}{\kappa}(2-D)(1-D)MR^{2-D} \\ & - \frac{e^2}{R^2}(\ddot{z}^\alpha \ddot{z}_\alpha + \epsilon a^2) + (3-D)(2-D)e^2[\epsilon a_1 R^{2-D} \\ & + aR^{5-2D} - aR^{1-D} - \epsilon Da^2 R^{2-D}] \end{aligned} \quad (61)$$

The flux of null fluid energy is then given by

$$N_f = - \int_{S^{D-2}} T_f{}^\alpha{}_\beta \dot{z}_\alpha n^\beta R^{D-3} d\Omega \quad (62)$$

and since  $T_f{}^\alpha{}_\beta = \rho \lambda^\alpha \lambda_\beta$  for the special case  $p = \Lambda = 0$  that we are examining, one finds that

$$N_f = \epsilon \epsilon_1 \int_{S^{D-2}} \rho R^{D-2} d\Omega \quad (63)$$

where  $\rho$  is given in (61). The flux of electromagnetic energy is similarly given by

$$N_e = - \int_{S^{D-2}} T_e{}^\alpha{}_\beta \dot{z}_\alpha n^\beta R^{D-3} d\Omega \quad (64)$$

and for the solution we are examining, one finds that

$$\begin{aligned}
-T_e{}^\alpha{}_\beta \dot{z}_\alpha n^\beta &= \epsilon_1 \epsilon (3-D) e [(3-D)e(A-\epsilon)R^{5-2D} \\
&\quad + \dot{e}R^{6-2D} + \epsilon(c_{,\alpha}\dot{z}^\alpha)R^{3-D}] + \epsilon\epsilon_1 R(c_{,\alpha}c'^\alpha) \\
&\quad + (3-D)^2 e^2 (R_{,\alpha}n^\alpha)R^{4-2D} \\
&\quad + (3-D)eR^{5-2D}\dot{e}(\lambda_\alpha n^\alpha) \\
&\quad + \epsilon(3-D)eR^{2-D}(c_{,\alpha}n^\alpha)
\end{aligned} \tag{65}$$

for the case  $c = c(\tau, a, a_k)$  and arbitrary. Taking the special case  $e = \text{constant}$ ,  $c = -ea$  of Bonnor-Vaidya [1], this simplifies and hence (64) becomes

$$N_e = \epsilon_1 e^2 \int_{S^{D-2}} d\Omega [a^2 + \epsilon(\dot{z}^\alpha \ddot{z}_\alpha)] R^{D-4}. \tag{66}$$

The total energy flux is given by

$$\begin{aligned}
N &= N_e + N_f = \epsilon_1 \int_{S^{D-2}} \left[ \frac{(2-D)\epsilon}{\kappa} \dot{M} + \frac{\epsilon}{\kappa} a(2-D)(1-D)M \right. \\
&\quad \left. + (3-D)(2-D)e^2 a_1 - D(2-D)(3-D)e^2 a^2 \right]
\end{aligned} \tag{67}$$

for  $R$  large enough. For a charge with acceleration  $|\ddot{z}_\alpha| = \kappa_1$ , we have (see the Appendix)

$$\begin{aligned}
N &= \frac{1}{2}(2-D)\{\dot{M}\epsilon + 2(3-D)e^2(\kappa_1)^2\Omega_{D-2} \\
&\quad - D(3-D)e^2(\kappa_1)^2\Omega_{D-3}\Gamma(\frac{D-2}{2})\gamma_D\}\epsilon_1,
\end{aligned} \tag{68}$$

where

$$\gamma_D = \frac{-2\sqrt{\pi}}{\Gamma(\frac{D-1}{2})} + 2^D \frac{\Gamma(\frac{D+2}{2})}{\Gamma(D)}.$$

Notice that when  $D = 4$ , this reduces to the result of Bonnor-Vaidya in Ref.[1]. We also give the  $D = 6$  case as another example.



$$N = \begin{cases} \epsilon_1 [-\epsilon \dot{M} - \frac{8\pi}{3} e^2 (\kappa_1)^2] & (D = 4) \\ \epsilon_1 [-2\epsilon \dot{M} - \frac{96\pi^2}{15} e^2 (\kappa_1)^2] & (D = 6) \end{cases} \quad (69)$$

**Remark 6.** To be consistent with Bonnor and Vaidya we take  $\epsilon_1 = -1$  which is also consistent with our Remark 1. Electromagnetic energy flux (66) is finite only when  $D = 4$ , but the total energy flux (68) is finite, for all  $D$  (due to the cancelation of divergent terms in  $N_e$  and  $N_f$ ).

**Remark 7.** The difference between the energy flux expressions in the  $D = 4$  flat space (29) with one obtained above (69) is due to the scaling factor ( $\frac{1}{4\pi}$ ). Apart from this difference the energy flux expressions obtained from classical electromagnetism and general relativity are exactly the same for the special case  $c = -ea$  and  $M = \text{constant}$ . For other choices we obtain different expressions for the energy flux. For  $D > 4$  since we do not have gauge equivalent solutions (see Remark 4) the energy flux expressions obtained from classical electromagnetism and general relativity are different.

For the  $D = 3$  case, substituting the special choice  $e = \text{constant}$ ,  $c = -ea$  of [1], the expression for  $\rho$  given in (57) reduces to

$$\begin{aligned} \rho = & -\frac{\dot{M}}{\kappa R} + \frac{2Ma}{\kappa R} - \frac{e^2}{R^2} (\ddot{z}^\alpha \ddot{z}_\alpha) - \epsilon \frac{e^2 a_1}{R} - \epsilon \frac{e^2 a^2}{R^2} \\ & - \frac{e^2 a}{R} \ln R + \frac{e^2 a}{R} [3\epsilon a + \frac{1}{R} - \frac{1}{2}]. \end{aligned} \quad (70)$$

Similarly the energy flux due to the fluid is found as

$$N_f = \epsilon \epsilon_1 \int_0^{2\pi} \rho R d\theta$$

where  $\rho$  is given in (70). For the energy flux due to the electromagnetic field, one finds

$$\begin{aligned}
-T_e{}^\alpha{}_\beta \dot{z}_\alpha n^\beta &= \epsilon_1 \epsilon \frac{e^2}{R} (A - \epsilon) + \epsilon \epsilon_1 e \dot{e} \ln R + \epsilon_1 e (c_{,\alpha} \dot{z}^\alpha) \\
&\quad + \epsilon \epsilon_1 R (c_{,\alpha} c'^\alpha) + \frac{e^2}{R^2} (R_{,\alpha} n^\alpha) + \frac{e \dot{e}}{R} \ln R (\lambda_\alpha n^\alpha) \\
&\quad + \epsilon \frac{e}{R} (c_{,\alpha} n'^\alpha)
\end{aligned} \tag{71}$$

Here  $c = c(\tau, a, a_k)$  is an arbitrary function of its arguments. For the special case  $e = \text{constant}$ ,  $c = -ea$  energy flux due to the electromagnetic field becomes

$$N_e = \frac{e^2}{R} \epsilon_1 \int_0^{2\pi} d\theta [a^2 + \epsilon (\ddot{z}^\alpha \ddot{z}_\alpha)]$$

The total energy loss is given by

$$\begin{aligned}
N = N_e + N_f = \epsilon_1 \int_0^{2\pi} d\theta & \left[ -\frac{\epsilon \dot{M}}{\kappa} + \frac{2\epsilon M a}{\kappa} - e^2 a_1 - \epsilon a e^2 \ln R \right. \\
& \left. + 3e^2 a^2 + \frac{\epsilon}{R} e^2 a - \frac{\epsilon}{2} e^2 a \right].
\end{aligned} \tag{72}$$

Assuming that the curve  $C$  is time like ( $\epsilon = -1$ ), performing the angular integration in (72), then taking  $R$  large enough we get

$$N = \left[ \frac{1}{2} \dot{M} + \pi e^2 \kappa_1^2 \right] \epsilon_1$$

**Remark 8.** The sign of the energy flux expression in three dimensions is opposite to the one obtained in other dimensions ( $D > 3$ ).

## 6 Conclusion

We have found exact solutions of the  $D$ -dimensional Einstein Maxwell field equations with a null perfect fluid source. Physically these solutions describe

the electromagnetic and gravitational fields of a charged particle moving on an arbitrary curve  $C$  in a  $D$ - dimensional manifold. The metric and the electromagnetic vector potential arbitrarily depend on a scalar,  $c(\tau_0, a, a_k)$  which can be related to the curvatures of the curve  $C$ . In four dimensions with a special choice of this scalar, our solution matches with the Bonnor-Vaidya metric [1]. For other choices we have different solutions. In higher dimensions our solutions given in Theorem 6, for all  $D$ , can be considered as the accelerated Tangherlini [15] solutions.

We have also studied the flux of electromagnetic energy due to the acceleration of charged particles. We observed that the energy flux formula, for all dimensions, depends on the choice of the scalar  $c$  in terms of the functions  $a, a_k$  (or the curvature scalars of the curve  $C$ ). In dimensions  $D > 4$  electromagnetic and fluid energy fluxes diverge for large values of  $R$  but the total energy flux is finite. We obtained the energy flux expression corresponding to a special choice,  $c = -ea$ , for all dimensions.

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## Appendix: Serret-Frenet Frames.

In this Appendix we shall give the Serret-Frenet frame in four dimensions which can be easily extended to any arbitrary dimension  $D$ . The curve  $C$  described in Section 2 has the tangent vector  $T^\mu = \dot{z}^\mu$ . Starting from this tangent vector, by repeated differentiation with respect to the arclength parameter  $\tau_0$ , one can generate an orthonormal frame  $\{T^\mu, N_1^\mu, N_2^\mu, N_3^\mu\}$ , the *Serret-Frenet frame*.

$$\dot{T}^\mu = \kappa_1 N_1^\mu, \quad (73)$$

$$\dot{N}_1^\mu = \kappa_1 T^\mu - \kappa_2 N_2^\mu, \quad (74)$$

$$\dot{N}_2^\mu = \kappa_2 N_1^\mu - \kappa_3 N_3^\mu, \quad (75)$$

$$\dot{N}_3^\mu = \kappa_3 N_2^\mu \quad (76)$$

Here  $\kappa_i$ , ( $i = 1, 2, 3$ ) are the curvatures of the curve  $C$  at the point  $z^\mu(\tau_0)$ . The normal vectors  $N_i$ , ( $i = 1, 2, 3$ ) are space like unit vectors. Hence at the point  $z^\mu(\tau_0)$  on the curve we have an orthonormal frame which can be used as a basis of the tangent space at this point. In Section 2 we have also defined some scalars

$$a_k = \frac{d^k \ddot{z}_\mu}{d\tau_0^k} \lambda^\mu$$

where  $\lambda_\mu = \epsilon T^\mu + \epsilon_1 \frac{n^\mu}{R}$ . Here  $n^\mu$  is a space like vector orthogonal to  $T^\mu$ . Hence we let

$$n^\mu = \alpha N_1^\mu + \beta N_2^\mu + \gamma N_3^\mu$$

with  $\alpha^2 + \beta^2 + \gamma^2 = R^2$ . One can choose the spherical angles  $\theta \in (0, \pi)$ ,  $\phi \in (0, 2\pi)$  such that

$$\alpha = R \cos \theta, \quad \beta = R \sin \theta \cos \phi, \quad \gamma = R \sin \theta \sin \phi$$

Hence we can calculate the scalars  $a_k$  in terms of the curvatures and the spherical coordinates  $(\theta, \phi)$  at the point  $z^\mu(\tau_0)$ . These expressions are quite useful in the energy flux formulas. As an example we give  $a$  and  $a_1$ .

$$a = -\epsilon \epsilon_1 \kappa_1 \cos \theta, \quad a_1 = (\kappa_1)^2 - \epsilon \epsilon_1 \dot{\kappa}_1 \cos \theta + \epsilon \epsilon_1 \kappa_1 \kappa_2 \sin \theta \cos \phi \quad (77)$$

All other  $a_k$ 's can be found similarly. Hence for all  $k$ , these scalars depend on the curvatures and the spherical angles.

In four dimensions the space like vector  $n$  is given by (we shall omit the indices on the vectors)

$$n = R [N_1 \cos \theta + N_2 \sin \theta \cos \phi + N_3 \sin \theta \sin \phi]$$

where  $\theta \in (0, \pi)$  and  $\phi \in (0, 2\pi)$ . The line element on  $S^2$  is

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2$$

The solid angle integral is

$$\int_{S^2} d\Omega = 4\pi.$$

In six dimensions we have

$$\begin{aligned} n = R [N_1 \cos \theta + N_2 \sin \theta \cos \phi_1 + N_3 \sin \theta \sin \phi_1 \cos \phi_2 \\ + N_4 \sin \theta \sin \phi_1 \sin \phi_2 \cos \phi_3 + N_5 \sin \theta \sin \phi_1 \sin \phi_2 \sin \phi_3] \end{aligned} \quad (78)$$

where  $\theta, \phi_1, \phi_2 \in (0, \pi)$  and  $\phi_3 \in (0, 2\pi)$  and

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi_1^2 + \sin^2 \theta \sin^2 \phi_1 d\phi_2^2 + \sin^2 \theta \sin^2 \phi_1 \sin^2 \phi_2 d\phi_3^2.$$

The solid angle integral is

$$\int_{S^4} d\Omega = \frac{8\pi^2}{3}.$$

The solid angle integral in  $D$ -dimensions is

$$\Omega_{D-2} \equiv \int_{S^{D-2}} d\Omega = \frac{(D-1) \pi^{(D-1)/2}}{\Gamma(\frac{D+1}{2})}.$$

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